Summing the series in parentheses we obtain

$$S = a\left\{ kn^2 - 2nk(k-1) + \frac{4k^3 - 6k^2 + 2k}{3} \right\} + b\left\{ kn^2 - 2nk^2 + \frac{4k^3 - k}{3} \right\},$$

which may be written as

$$(a+b)kn^2 - 2nk(ak + bk - a) + \left\{ \frac{4}{3}k^3(a+b) - 2ak^2 + \frac{k}{3}(2a-b) \right\}.$$

Also solved by H. C. Feemster, Horace Olson and the Proposer.

441. Proposed by W. D. Cairns, Oberlin College.

Prove that the equation $(e-1)x = e^x - 1$ has two and only two real roots.

I. Solution by H. S. Uhler, Yale University.

Let $y = e^x - 1 - (e-1)x$ and observe that $y = 0$ for $x = 0$ and $x = 1$. It remains to show that there can be no more real roots.

$$\frac{dy}{dx} = e^x - e + 1, \quad (1) \qquad \frac{d^2y}{dx^2} = e^x. \quad (2)$$

Equation (1) shows that the slope of the tangent is positive for all values of x greater than $\log_e(e-1)$ and negative for all values of x less than this value ($x_0 = 0.541325$). Since e^x is essentially positive, equation (2) indicates formally that the single stationary point, indicated by equation (1), corresponds to a minimum value of y. The coördinates of the minimum are $x_0 = \log_e(e-1)$ and $y_0 = (e-1)(1-x_0) - 1 = -0.211867$. It is clear, therefore, from the properties of the graph that the curve cannot cut the axis of x in more than two points and hence the given equation has two and only two real roots.

II. Solution by Grace M. Bareis, Ohio State University.

Writing e in series form in each member and transposing, the equation becomes

$$x\left[\frac{x-1}{\lfloor 2} + \frac{x^2-1}{\lfloor 3} + \frac{x^3-1}{\lfloor 4} + \cdots\right] = 0, \quad \text{or} \quad x(x-1)\left[\frac{1}{\lfloor 2} + \frac{x+1}{\lfloor 3} + \frac{x^2+x+1}{\lfloor 4} + \cdots\right] = 0.$$

Hence, $x = 0$, $x - 1 = 0$, or $\dfrac{1}{\lfloor 2} + \dfrac{x+1}{\lfloor 3} + \dfrac{x^2+x+1}{\lfloor 4} + \cdots = 0$. Since each term of the left member of the last equation is positive, this equation can have no real positive root.

To prove that the original equation can have no real negative root, put it in the form

$$x = \frac{e^x - 1}{e - 1},$$

or $x = e^{x-1} + e^{x-2} + e^{x-3} + \cdots$, a convergent series. Any real negative number makes the left member negative but the right member positive, and hence there are no real negative roots. Hence, 0 and 1 are the only real roots.

Solutions were also received from W. L. Agard, F. L. Griffin, H. C. Feemster, Walter C. Eells, Horace Olson, C. E. Horne, Irby C. Nichols, and Frank Irwin.

GEOMETRY.

467. Proposed by E. T. Bell, Seattle, Washington.

It is well-known that if i, j, k, l are concyclic points, W_i the Wallace line (frequently, and erroneously, called the Simson line), of i with respect to the triangle jkl, then W_i, W_j, W_k, W_l are concurrent, say in the point $\{i, j, k, l\}$. If 1, 2, 3, $\cdots$ denote concyclic points, prove that:

(i) $\{1, 2, 3, 4\}$, $\{1, 2, 3, 5\}$, $\{1, 2, 4, 5\}$, $\{1, 3, 4, 5\}$, $\{2, 3, 4, 5\}$ are concyclic; say on the circle [1, 2, 3, 4, 5];

(ii) Starting with 1, 2, 3, 4, 5, 6, omitting each point in turn, by (i), six circles, are found; these are concurrent, say in the point $\{1, 2, 3, 4, 5, 6\}$;

(iii) Starting with 1, 2, 3, 4, 5, 6, 7, seven points of the kind in (ii) are found; these lie on a circle.

(iv) Continuing thus indefinitely, there is, in each case, finally a unique point or circle according as the number of initial points is even or odd. Also, at any stage, the point of concurrence on the circle bears a simple relation to the initial points: what is it?

I. Solution by J. W. Clawson, Collegeville, Pa.

If H_i is the orthocenter of triangle jkl, O the center of the circle, radius R, on which the points i, j, k, l lie,—

iH_j = twice distance of O from $kl = jH_i$. Also iH_j is parallel to jH_i; hence H_iH_j is equal to, and parallel to, ij. It follows that $H_iH_jH_kH_l$ is inversely congruent with $ijkl$, the corresponding sides of the quadrilaterals being equal and parallel, but arranged in opposite orders.

Again, iH_i, jH_j, kH_k, lH_l have a common mid-point; and, since iH_i bisects W_i and so on, this common point is the point of concurrency of W_i, W_j, W_k, W_l. The point of concurrency $\{i, j, k, l\}$ bisects the join of O to the center of the circle circumscribing $H_iH_jH_kH_l$.

(i) If H_{123} is the orthocenter of triangle 123, H_{123}, H_{234}, H_{341}, H_{412} are four vertices of a pentagon inversely congruent with 12345. Moreover the sides of this pentagon are parallel to the corresponding sides of the pentagon 12345. Call the center of the circumcircle of this pentagon, P_5. Similarly H_{123}, H_{235}, H_{351}, H_{512} and three other sets of orthocenters determine four other congruent pentagons with parallel sides. Each of the orthocenters occurs as a vertex of two of the pentagons. It follows that P_5, P_4, P_3, P_2, P_1, the centers of the circles circumscribing these five pentagons, form a pentagon which is congruent with each of the pentagons and hence inversely congruent with the original pentagon 12345. The points P_5, P_4, P_3, P_2, P_1 are therefore concyclic. Call the center of their circle, O_6.

It follows at once that the points (1234), (2345), (3451), (4512), (5123) are concyclic on a circle of half the radius of the given circle, its center bisecting the join of O to O_6.

(ii) The circle, center O_6, passes through P_5, the center of a circle determined by the orthocenters of the triangles obtained by selecting triads of points from 1, 2, 3, 4. In the same way if we consider the five points 1, 2, 3, 4, 6 we shall obtain a circle, center O_5, which passes through P_5 and through four new points. In this way we can get six hexagons, one of which has P_1, P_2, P_3, P_4, P_5 for five of its vertices, each inversely congruent with the hexagon 123456 and five vertices of each hexagon (orthocenters of triangles obtained by selecting three points from the six) belonging also to a different hexagon of the system. It follows that $O_6O_5O_4O_3O_2O_1$ forms a hexagon which is congruent with each of the hexagons and hence is inversely congruent with the original hexagon 123456. Hence $O_6, O_5, O_4, O_3, O_2, O_1$ are concyclic; and the circles with $O_6, O_5, O_4, O_3, O_2, O_1$ as centers and with R as radius concur at a point, say Q_7.

It follows at once that circles (1, 2, 3, 4, 5), $\cdots$ concur at a point halfway between Q_7 and O.

(iii) This process can be continued indefinitely as stated in the problem.

II. Solution by Norman Anning, Chilliwack, B. C.

Letting $e^{i\phi} = \cos\phi + i\sin\phi$, then $Re^{i\theta_j}$ $(j = 1, 2, 3, 4 \cdots)$ are points on a circle of radius R, whose center is at the origin. The W line of 4 quâ 123 passes through the foot of the perpendicular from 4 on line 12 and makes with this perpendicular an angle equal to that between 13 and 14. All points on W_4 are given by

$$Re^{i\theta_4} + ke^{i\frac{\theta_1+\theta_2}{2}} + xe^{i\frac{\theta_1+\theta_2+\theta_3-\theta_4}{2}},$$

where x is a running coördinate on the line and k, the distance from 4 to 12, is to be determined. Projecting on the radius vector midway between 1 and 2,

$$k + R\cos\left(\theta_4 - \frac{\theta_1+\theta_2}{2}\right) = R\cos\frac{\theta_1-\theta_2}{2}, \qquad k = 2R\sin\frac{\theta_4-\theta_2}{2}\sin\frac{\theta_4-\theta_1}{2}.$$

$$W_4 \text{ is } Re^{i\theta_4} + 2R\sin\frac{\theta_4-\theta_2}{2}\sin\frac{\theta_4-\theta_1}{2}e^{i\frac{\theta_1+\theta_2}{2}} + xe^{i\frac{\theta_1+\theta_2+\theta_3-\theta_4}{2}}.$$

$$W_3 \text{ is } Re^{i\theta_3} + 2R\sin\frac{\theta_3-\theta_2}{2}\sin\frac{\theta_3-\theta_1}{2}e^{i\frac{\theta_1+\theta_2}{2}} + ye^{i\frac{\theta_1+\theta_2-\theta_3+\theta_4}{2}}.$$

To find the common point of these lines equate the two expressions, separate reals and imaginaries and solve for x and y. Result:

$$x = y = 2R\cos\frac{\theta_4 + \theta_3 - \theta_2 - \theta_1}{2}.$$

When the exponential values are inserted for the sines and cosines, the expression for the common point of W_4 and W_3 becomes

$$\frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4}).$$

The symmetry of this expression shows that the point is the common point of the four W lines.

$$\{1, 2, 3, 4\} \text{ is } \frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5} - e^{i\theta_5}),$$

$$\{1, 2, 3, 5\} \text{ is } \frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5} - e^{i\theta_4}),$$

$$\cdots\cdots\cdots\cdots$$

$$\{2, 3, 4, 5\} \text{ is } \frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5} - e^{i\theta_1}).$$

These 5 points lie on the circle whose center is the point, $\frac{R}{2}\sum_{p=1}^{p=5} e^{i\theta_p}$, and whose radius is $R/2$. Where ϕ is variable, the circle

$$[1, 2, 3, 4, 5] \text{ is } \frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5}) + \frac{R}{2}e^{i\phi},$$

$$[1, 2, 3, 4, 6] \text{ is } \frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_6}) + \frac{R}{2}e^{i\phi},$$

$$\cdots\cdots\cdots\cdots$$

$$[2, 3, 4, 5, 6] \text{ is } \frac{R}{2}(e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5} + e^{i\theta_6}) + \frac{R}{2}e^{i\phi}.$$

These 6 circles have the common point,

$$\frac{R}{2}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} + e^{i\theta_4} + e^{i\theta_5} + e^{i\theta_6}).$$

In the next step the 7 points, of which $\{1, 2, 3, 4, 5, 6\}$ is one, lie on the circle whose center is $\frac{R}{2}\sum_{p=1}^{p=7} e^{i\theta_p}$ and whose radius is $R/2$. Then follows a group of 8 such circles having the common point, $\frac{R}{2}\sum_{p=1}^{p=8} e^{i\theta_p}$, etc., etc.

In general, with an even number of initial points this method yields a point. The vector to this point from the origin is half the sum of the vectors to the initial points. With an odd number of initial points we have a circle whose radius is half that of the original circle and the vector to whose center is half the sum of the vectors to the initial points. The "W" points on this circle are in perspective relation to the initial points.

CALCULUS.

363. Proposed by B. F. FINKEL, Drury College.

The axis of a right prism whose cross-section is a regular polygon of n sides coincides with the diameter of a sphere of radius R. Find the surface of the sphere included within the prism.

I. SOLUTION BY H. S. UHLER, Yale University.

Let a denote the apothem of the right-section of the prism. The required area will be expressed as a function of a, n, and R. This area may be analyzed as $4n$ triangles on the surface of